## 1. PHD PROJECT DESCRIPTION

**Project title:**

**Geometric and dynamical analysis of flows in selected fluid mechanics models**

**Project goals**

- Understand local approximation of contour dynamics for 2D Euler and generalized surface quasi-geostrophic equation (gSQG)
- Determine class of finite-time singularities developed by a geometric flow approximating the contour dynamics corresponding to gSQG.
- Establish the existence and describe the qualitative properties of periodic solutions for the forced gSQG equations.

**Outline**

The basis of our considerations is the generalized surface quasi geostrophic equation (gSQG) $\partial_t\theta + u\cdot\nabla\theta = 0$, $u = -\nabla^{\perp}(-\Delta)^{-1+\alpha}\theta$, where $\alpha \in [0,1)$ is a parameter. In particular for $\alpha = 0$, the equation becomes the vorticity formulation of the two dimensional incompressible Euler equation, which describes the evolution of inviscid, incompressible fluids with constant density, where the motion of fluid particles is driven by internal pressure [6], [7]. Recent studies for the gSQG equation have focused on patch-type solutions—those that remain characteristic functions of bounded regions in the Cartesian plane for all times. Particular examples are well-known V-states: time-periodic convex bounded regions that rotate around their center of mass with a constant angular velocity.

We know that the evolution of the boundary of general patches is governed by a non-local flow, called contour dynamic equation. The nonlocality introduces significant analytical challenges, as the velocity field evolving the boundary, depends on the global geometry of the patch. Understanding whether singularities can form in finite time, as well as describing the long-term behavior and possible self-similar structures, are central questions in this context.

In the case of the 2D Euler equation, a geometric flow of planar curves was proposed by Goldstein and Petrich as a formal approximation of the contour dynamics equation, providing an alternative framework for the local analysis of patch evolution. An important property is the fact that corresponding flow of scalar curvatures satisfies focusing modified Korteweg-de Vries equation. Recently, it was shown that an arbitrary double spiral is a finite-time singularity developed by a self-similar solution of the geometric flow [3], [5]. Moreover, the curvatures corresponding to this self-similar solution tends to a distribution in the form of linear combination of Dirac delta and Cauchy principal value [4].

The project is supervised by two researchers: A. Ćwiszewski and P. Kokocki, closely collaborating on the field of partial differential equations. One of tasks of this doctoral research project is to formally derive a geometric flow that approximates the contour dynamics of patch-type solutions to the generalized SQG equation. Then our subsequent problem is to identify and classify singularities developed by self-similar solutions of the geometric flow. We are also interested in analyzing their geometrical structure by studying singularities developed by the corresponding curvature flow. In particular, we aim to characterize the mechanisms that lead to singularity formation and investigate conditions under which such singularities emerge in finite time. This will enhance understanding of the dynamics of generalized gSQG models by employing simplified geometric approximations.

Beyond patch evolution, we are also interested in other dynamical properties of gSQG equations. By augmenting the equation with periodically varying forcing terms, we will investigate the existence and bifurcations of time-periodic solutions other than the V-states mentioned above. Our approach relies on the application of fixed-point methods and topological tools, including Leray–Schauder degree theory [1], [2], the Banach fixed-point theorem, and the Crandall–Rabinowitz bifurcation theorem.

**Work plan**

- Derive the geometric flow that formally approximates the contour dynamics of the patch-type solutions for the gSQG equations and investigate its properties.
- Identify the form of self-similar solutions of the geometric flow and derive the corresponding profile equations. Study the associated curvature flow to understand the geometric structure of these solutions.
- Classify the types of finite-time singularities and characterize their geometric features.
- Apply topological tools to prove the existence of time-periodic solutions to forced gSQG equations.

**Literature**

[1] A. Ćwiszewski, Degree theory for perturbations of m -accretive operators generating compact semigroups with constraints, J. Evol. Equ. 7 (2007), no. 1, 1–33.
[2] A. Ćwiszewski, P. Kokocki, Periodic solutions of nonlinear hyperbolic evolution systems, J. Evol. Equ. 10 (2010), no. 3, 677–710.
[3] K. Dunst, P. Kokocki, Double spiral singularities for a flow related with the 2D Euler equation, SIAM Journal on Mathematical Analysis, vol. 53 (2021), no. 4, 4727-4743.
[4] K. Dunst, P. Kokocki, On global solutions of defocusing mKdV equation with specific initial data of critical regularity, Physica D: Nonlinear Phenomena, vol. 417 (2021), art. no. 132810.
[5] P. Kokocki, Total integrals of Ablowitz-Segur solutions for the inhomogeneous Painlevé II equation, Studies in Applied Mathematics, vol. 144 (2020), no. 4, 504-547.
[6] A.J. Majda, A.L. Bertozzi, Vorticity and incompressible flow, Cambridge Texts in Applied Mathematics, 27. Cambridge University Press, Cambridge, 2002.
[7] C. Marchioro, M. Pulvirenti, Mathematical theory of incompressible nonviscous fluids. Springer, New York, 1994.

**Required initial knowledge and skills of the PhD candidate**

- Analytical thinking.
- Willingness to self-study.
- Understanding of mathematical analysis.
- Basic knowledge of functional analysis, complex analysis, topology and partial differential equations.

**Expected development of the PhD candidate's knowledge and skills**

- Advanced skills in functional analysis, partial differential equations, topology, with the ability to apply these techniques in partial differential equations.
- Understanding of the physical interpretation of the mathematical models under study and their real-world applications.
- An enhanced ability to conduct independent, original research, and formulate mathematical hypotheses.
- Communication and collaboration skills through presenting research results, writing papers, and collaborating with the research environment.