PHD PROJECT DESCRIPTION

Project title: Geometric analysis of flow in selected fluid mechanics models

1. Project goals

The main objective of the project is to investigate selected equations of fluid mechanics using Arnold’s geometric framework, along the following directions.

- Establish a geometric formulation of selected fluid models, with an initial focus on the Lagrangian Averaged Euler equations (LAE), which arise as a regularization of the classical incompressible Euler system.
- Characterize the analytical properties of the corresponding geodesic flows, including the regularity and Fredholm properties of the associated exponential maps, as well as the existence and stability of conjugate points.
- Determine in what sense the geometric structures of the LAE equations degenerate to those of the classical Euler flow as the averaging parameter tends to zero.

2. Outline

The geometric formulation of hydrodynamics, initiated by Arnold (1966), states that the dynamics of an ideal incompressible fluid correspond to geodesic flow on the infinite-dimensional group of volume-preserving diffeomorphisms. In this framework, the Euler equations arises as the geodesic flow associated with a right-invariant L^2 metric, meaning that the corresponding fluid trajectories follow the straightest possible paths in the configuration space. The stability of fluid motion and the emergence of turbulence are closely related to the curvature and the presence of conjugate points on the underlying manifold; their rigorous study in this infinite-dimensional setting was carried out by Misiołek (1993).

This project aims to apply these ideas to selected models of fluid mechanics, in particular the Lagrangian Averaged Euler equations (LAE), which can be viewed as a geometric regularization of the classical incompressible Euler system. Within Arnold's geometric framework, the fluid models under consideration are interpreted as geodesic flows on the infinite-dimensional group of volume-preserving diffeomorphisms equipped with an appropriate Riemannian metric. The project focuses on the analytical properties of these flows, with particular attention to the behavior of the exponential map, its regularity, and its Fredholm properties. Stability phenomena are further investigated through the study of conjugate points, which allows one to relate qualitative features of fluid motion to curvature properties of the underlying infinite-dimensional manifold. Special attention is given to the dependence of these geometric structures on the averaging parameter, and in particular to the limiting behavior as the LAE equations converge to the classical Euler system.

3. Work plan

- Study the necessary background in functional analysis and differential geometry, with particular focus on geodesic flows, exponential maps, and their analytical properties on the groups of diffeomorphisms.

- Formulate selected fluid models, with an initial focus on the Lagrangian Averaged Euler equations, within the geometric framework. Construct the associated Riemannian structures on the group of volume-preserving diffeomorphisms.
- Derive the corresponding geodesic equations. Investigate Fredholm properties of the corresponding exponential map and analyze the existence and stability of conjugate points along geodesics using Jacobi field equations.
- Investigate the limiting behavior of the LAE equations as the averaging parameter tends to zero, and determine in what sense the corresponding conjugate points converge to their classical Euler counterparts.

4. Literature

[1] V. Arnold, B. Khesin, *Topological Methods in Hydrodynamics*, 2nd ed. Springer, Cham, 2021.
[2] M. P. do Carmo, *Riemannian Geometry*, Birkhäuser, Boston, 1992.
[3] D. D. Holm, J. E. Marsden, and T. S. Ratiu, *Euler–Poincaré equations and semidirect products with applications to continuum theories*, Adv. in Math., vol. 137 (1998) p. 1–81.
[4] A.J. Majda, A.L. Bertozzi, Vorticity and incompressible flow, Cambridge Texts in Applied Mathematics, 27. Cambridge University Press, Cambridge, 2002.
[5] G. Misiołek, Stability of flows of ideal fluids and the geometry of the group of diffeomorphisms, Indiana University Mathematics Journal 42 (1993), no. 1, 215–235.
[6] S. Shkoller, *The Lagrangian Averaged Euler (LAE-α) Equations with Free-Slip or Mixed Boundary Conditions,* Geometry, Mechanics, and Dynamics. Springer, New York, 2002.

5. Required initial knowledge and skills of the PhD candidate

- Analytical thinking.
- Willingness to self-study.
- Understanding of mathematical analysis.
- Basic knowledge of functional analysis, differential geometry.

6. Expected development of the PhD candidate’s knowledge and skills

- Advanced skills in functional analysis, differential geometry and topology, with the ability to apply these techniques to partial differential equations.
- Understanding of the physical interpretation of the fluid models under study and their real-world applications.
- An enhanced ability to conduct independent, original research and to formulate mathematical hypotheses.
- A developed ability to connect geometric structures with physical models of fluid mechanics.
- Communication and collaboration skills developed through presenting research results, writing scientific papers, and engaging with the broader research community..